

# Adaptive Entanglement Purification Protocols with Two-way Classical Communication

Alan W. Leung\* and Peter W. Shor†

*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

(Dated: February 1, 2008)

We present a family of entanglement purification protocols that generalize four previous methods, namely the recurrence method, the modified recurrence method, and the two methods proposed by Maneva-Smolín and Leung-Shor. We will show that this family of protocols have improved yields over a wide range of initial fidelities  $F$ , and hence imply new lower bounds on the quantum capacity assisted by two-way classical communication of the quantum depolarizing channel. In particular, the yields of these protocols are higher than the yield of universal hashing for  $F$  less than 0.993 and as  $F$  goes to 1.

PACS numbers: 03.67.Hk

## I. INTRODUCTION

Quantum information theory studies the information processing power one can achieve by harnessing quantum mechanical principles. Many important results such as quantum teleportation, superdense coding, factoring and search algorithms make use of quantum entanglements as fundamental resources[2, 3, 4, 5]. Pure-state entanglements are therefore useful; however, when they are exposed to noise, they become mixed entangled states. It is thus important to study the procedures by which we can extract pure-state entanglements from mixed entangled states, and we call these procedures entanglement purification protocols(EPP). The present work in particular studies the scenario where the two parties - whom we call Alice and Bob throughout - are allowed to communicate classically. We will follow the framework of [6, 7, 8] and generalize these results to obtain a family of protocols with improved yields.

## II. ADAPTIVE ENTANGLEMENT PURIFICATION PROTOCOLS (AEPP)

### A. Notations

We denote von Neumann entropy by  $S(\rho)$ , Shannon entropy by  $H(p_0, p_1, \dots)$  and label the four Bell states with two classical bits  $(a, b)$  as follows:

$$\begin{aligned} 00 : |\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \\ 01 : |\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ 10 : |\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle) \\ 11 : |\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \end{aligned}$$

This work concerns the purification of the generalized Werner state[10],

$$\rho_F = F |\Phi^+\rangle \langle \Phi^+| + \frac{1-F}{3} \left( |\Phi^-\rangle \langle \Phi^-| + |\Psi^+\rangle \langle \Psi^+| + |\Psi^-\rangle \langle \Psi^-| \right).$$

At the beginning of these entanglement purification protocols, two persons, Alice and Bob, share a large number of quantum states  $\rho_F$ , say  $\rho_F^{\otimes N}$ , and they are allowed to communicate classically, apply unitary transformations and perform projective measurements. We place no restriction on the size of their ancilla systems so that we lost no generality in restricting their local operations to unitaries and projective measurements. In the end, the quantum states  $\Upsilon$  shared by Alice and Bob are to be a close approximation of the maximally entangled states  $(|\Phi^+\rangle \langle \Phi^+|)^{\otimes M}$ , or more precisely we require the fidelity between  $\Upsilon$  and  $(|\Phi^+\rangle \langle \Phi^+|)^{\otimes M}$  approaches zero as  $N$  goes to infinity. We define the yield of such protocols to be  $M/N$ .

We will often use the BXOR operation by which we mean the bilateral application of the two-bit quantum XOR (or controlled-NOT). We only consider the scenario in which Alice and Bob share two (or more) bipartite quantum states that are Bell diagonal and they apply BXOR to two pairs of quantum states such that one pair is the “source” and one pair is the “target”. Using the two classical bit notations, we write

$$\begin{aligned} BXOR(i, j) : \{0, 1\}^N &\rightarrow \{0, 1\}^N \\ (a_i, b_i) &\mapsto (a_i \oplus a_j, b_i) \\ (a_j, b_j) &\mapsto (a_j, b_i \oplus b_j) \\ (a_k, b_k) &\mapsto (a_k, b_k) \text{ if } k \neq i, j \end{aligned}$$

when Alice and Bob share  $N$  pairs of bipartite quantum states and they apply BXOR to the  $i$ th pair as source and the  $j$ th pair as target.

\*Electronic address: leung@math.mit.edu

†Electronic address: shor@math.mit.edu

## B. Description of AEPP

1. AEPP(a,2): Alice and Bob put the bipartite quantum states  $\rho_F^{\otimes N}$  into groups of two, apply BXOR(1,2)

$$(a_1, b_1, a_2, b_2) \mapsto (a_1 \oplus a_2, b_1, a_2, b_1 \oplus b_2)$$

and take projective measurements on the second pair along the z-axis. Using two-way classical communication channel, they can compare their measurement results. If the measurement results agree ( $b_1 \oplus b_2 = 0$ ), then it is likely that there has been no amplitude error and Alice and Bob will perform universal hashing on the first pair; if the results disagree ( $b_1 \oplus b_2 = 1$ ), they throw away the first pair because it is likely that an amplitude error has occurred. We give a graphical representation of this protocol in fig.1.

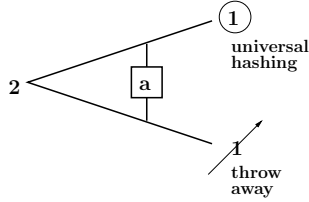


FIG. 1: AEPP(a,2)

2. AEPP(a,4): Alice and Bob put the bipartite quantum states  $\rho_F^{\otimes N}$  into groups of four, apply BXOR(1,4), BXOR(2,4), BXOR(3,4)

$$(a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4) \mapsto (a_1 \oplus a_4, b_1, a_2 \oplus a_4, b_2, a_3 \oplus a_4, b_3, a_4, b_1 \oplus b_2 \oplus b_3 \oplus b_4)$$

and take projective measurements on the fourth pair along the z-axis. Using two-way classical communication channel, they can compare their measurement results. If the measurement results agree ( $b_1 \oplus b_2 \oplus b_3 \oplus b_4 = 0$ ), then it is likely that there has been no amplitude error and Alice and Bob will perform universal hashing on the first three pairs together.

On the other hand, if the results disagree ( $b_1 \oplus b_2 \oplus b_3 \oplus b_4 = 1$ ), it is likely that there is one amplitude error and Alice and Bob want to locate this amplitude error. They do so by applying BXOR(2,1)

$$(a_1 \oplus a_4, b_1, a_2 \oplus a_4, b_2, a_3 \oplus a_4, b_3, a_4, 1) \mapsto (a_1 \oplus a_4, b_1 \oplus b_2, a_1 \oplus a_2, b_2, a_3 \oplus a_4, b_3, a_4, 1) \quad (1)$$

and taking projective measurements on the first pair along the z-axis. Note that the second pair ( $a_1 \oplus a_2, b_2$ ) and the third pair ( $a_3 \oplus a_4, b_3$ ) are no longer entangled. Alice and Bob then use classical communication channel to compare their results. If the results agree ( $b_1 \oplus b_2 = 0$ ),

then the amplitude error detected by the first measurements is more likely to be on either the third or the fourth pair than on the first two. Therefore Alice and Bob perform universal hashing on the second pair and throw away the third pair. If the results disagree ( $b_1 \oplus b_2 = 1$ ), then the amplitude error is more likely to be on the first two pairs. In this case, Alice and Bob perform universal hashing on the third pair and throw away the second pair.

Note that the amplitude error could have been on the fourth pair but this protocol works well even if that is the case; and also that with this procedure we always end up with one pair on which Alice and Bob can perform universal hashing when the first measurement results disagree. We represent this protocol graphically in fig.2.

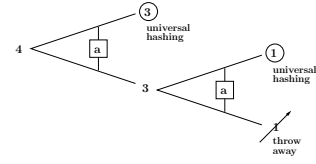


FIG. 2: AEPP(a,4)

3. AEPP(a,8): Alice and Bob put the bipartite quantum states  $\rho_F^{\otimes N}$  into groups of eight, apply BXOR(1,8), BXOR(2,8), BXOR(3,8), ..., BXOR(7,8)

$$(a_1, b_1, a_2, b_2, \dots, a_7, b_7, a_8, b_8) \mapsto (a_1 \oplus a_8, b_1, a_2 \oplus a_8, b_2, \dots, a_7 \oplus a_8, b_7, a_8, b_1 \oplus \dots \oplus b_8)$$

and take projective measurements on the eighth pair along the z-axis. Using classical communication channel, Alice and Bob compare their measurement results. If the results agree ( $b_1 \oplus \dots \oplus b_8 = 0$ ), then an amplitude error is not likely and they perform universal hashing on the first seven pairs together.

On the other hand, if the measurement results disagree ( $b_1 \oplus \dots \oplus b_8 = 1$ ), then Alice and Bob want to catch this amplitude error and they do that by applying BXOR(2,1), BXOR(3,1), BXOR(4,1)

$$(a_1 \oplus a_8, b_1, a_2 \oplus a_8, b_2, \dots, a_7 \oplus a_8, b_7, a_8, 1) \mapsto (a_1 \oplus a_8, b_1 \oplus b_2 \oplus b_3 \oplus b_4, a_1 \oplus a_2, b_2, a_1 \oplus a_3, b_3, a_1 \oplus a_4, b_4, a_5 \oplus a_8, b_5, a_6 \oplus a_8, b_6, a_7 \oplus a_8, b_7, a_8, 1)$$

and taking projective measurements on the first pair along the z-axis. Note that the second, third and fourth pairs are not entangled with the fifth, sixth and seventh pairs. After Alice and Bob compare their results with classical communication channel and if the results disagree ( $b_1 \oplus b_2 \oplus b_3 \oplus b_4 = 1$ ), they perform universal hashing on the fifth, sixth and seventh pairs because  $b_1 \oplus b_2 \oplus b_3 \oplus b_4 = 1$  and  $b_1 \oplus \dots \oplus b_8 = 1$  together imply  $b_5 \oplus b_6 \oplus b_7 \oplus b_8 = 0$ . The first four pairs are now represented by  $(a_1 \oplus a_8, 1, a_1 \oplus a_2, b_2, a_1 \oplus a_3, b_3, a_1 \oplus a_4, b_4)$ ,

and it can be easily seen that we are in the same situation as the left hand side of equation (1): Alice and Bob know that  $b_1 \oplus b_2 \oplus b_3 \oplus b_4 = 1$  and the pair on which they measured to find out this information has its phase error added to the other three pairs. Therefore Alice and Bob can apply the same procedure as equation (1) and end up with one pair that they will perform universal hashing on. Now if the results actually agree ( $b_1 \oplus b_2 \oplus b_3 \oplus b_4 = 0$ ), the same procedure still applies but we need to switch the roles played by the first four pairs and by the last four pair. We represent this protocol graphically in fig.3.

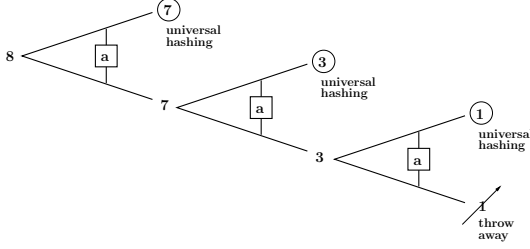


FIG. 3: AEPP(a,8)

4. AEPP(a,  $N=2^n$ ) and AEPP(p,  $N=2^n$ ): Clearly, the above procedures generalize to AEPP(a,  $N=2^n$ ) and can be proved inductively. The procedures - AEPP(a,  $N=2^n$ ) - we discussed so far focus on amplitude error. If we instead try to detect phase error by switching the source pairs and target pairs in all the BXOR operations and measuring along the x-axis rather than the z-axis, AEPP(p,  $N=2^n$ ) can be defined analogously. We represent the protocols AEPP(p,  $N=2^n$ ) graphically in fig.4, and we present the yields of AEPP(a,  $N=2^n$ ) for  $n = 2, 3, 4, 5, 6$  in fig.5.

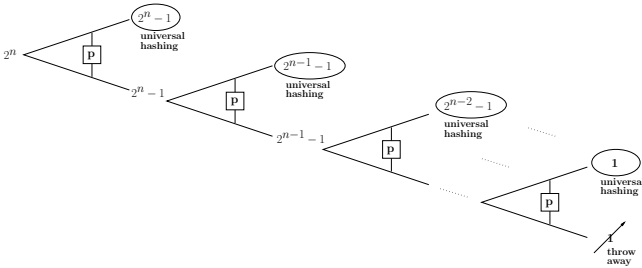


FIG. 4: AEPP(p,  $N=2^n$ )

### C. Generalization of previous methods

In this section, we show that four previous protocols - the recurrence method, the modified recurrence method and the two methods proposed by Maneva-Smolín and Leung-Shor - all belong to the family AEPP(a/p,  $N=2^n$ ).

1. The Recurrence Method: The recurrence method[6] is the repeat applications of AEPP(a,2). When Alice

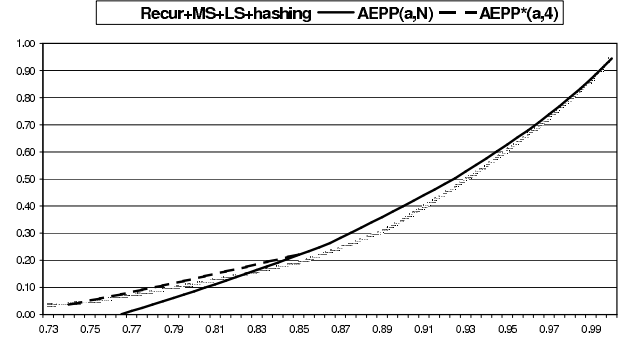


FIG. 5: Comparison of AEPP and previous methods: The lightly colored line is the yield of the four methods discussed in sec.II B; the solid line represents the yields of AEPP(a,  $N=2^n$ ) where  $n = 2, 3, 4, 5, 6$ ; the dashed line represents the optimized AEPP(a,4), which is denoted by AEPP\*(a,4)(see sec.II C and [9] for details)

and Bob have identical measurement results, rather than applying universal hashing right away, they repeatedly apply AEPP(a,2) until it is more beneficial to switch to hashing.

2. The modified recurrence method: The modified recurrence method[6] is the repeat, alternate applications of AEPP(a,2) and AEPP(p,2). After Alice and Bob apply AEPP(a,2) and obtain identical measurement results, they repeatedly and alternately apply AEPP(p,2), AEPP(a,2) and so forth until it becomes more beneficial to switch to universal hashing.

3. The Maneva-Smolín method: The Maneva-Smolín method[7] is to apply the first step of AEPP(a,  $N$ ). Perform universal hashing on the  $N-1$  pairs if the measurement results agree but throw away all the  $N-1$  pairs if they do not. This is illustrated in fig.6.

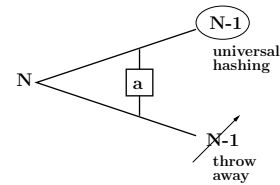


FIG. 6: The Maneva-Smolín method[7]

4. The Leung-Shor method: The Leung-Shor method[8] is a combination of the first step AEPP(a,4) and AEPP(p,4); however, this method fails to utilize all entanglements by throwing away the 3 pairs if the first measurement results disagree. This is illustrated in fig.7.

### D. Optimization

After we apply AEPP(a,  $N=2^n$ ), we might end up with either  $2^n - 1$  pairs or  $n-1$  groups of pairs ( $2^{n-1} - 1, 2^{n-2} - 1, \dots, 2^k - 1, \dots, 3$  and 1) pairs depending on the results of

